SPONTANEOUS BREAKING IN SUPERSYMMETRY

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ABSTRACT

Spontaneous symmetry breaking is considered for supersymmetric quantum mechanics and supersymmetric field theory.

1. INTRODUCTION

For our purposes a supersymmetric system (e.g. a quantum theory, a group representation) is one for which[1)]

(i) The underlying function space (usually a Hilbert space) $\mathcal{H}$ splits into two parts $\mathcal{H}_+ + \mathcal{H}_-$ (which may be thought of as bosonic and fermionic respectively) characterized by an operator σ such that $\sigma\mathcal{H}_\pm = \pm\mathcal{H}_\pm$.

(ii) There exist a finite number of (self-adjoint) operators Q_i (supersymmetric generators) which anti-commute with σ,

$$\{Q_i , \sigma\} = 0 , \qquad (1.1)$$

and

(iii) The operator of interest for the system (e.g. Hamiltonian or Laplacian) is a sum of squares of the Q_i, and commutes with each Q_i i.e.

$$H = \sum_i Q_i^2 , \qquad [H,Q_i] = 0. \qquad (1.2)$$

It is easy to see from (1.1) and (1.2) that the operator H commutes with σ. From this and (1.1) one sees that the eigenspaces of H with zero and non-zero eigenvalues behave quite differently. Since

$$H\psi_0 = 0 \quad \Leftrightarrow \quad Q_i\psi_0 = 0 \quad (i=1...n) , \qquad (1.3)$$

one sees that the eigenstates with eigenvalue zero, and only these eigenstates, are symmetric with respect to the Q_i. Further, since the Q_i and σ anti-commute, any eigenspace ψ_ε of H with non-zero eigenvalue ε must consist of two non-zero components $\psi_\varepsilon^\pm$ corresponding to $\sigma = \pm 1$ and thus the eigenspaces of H with non-zero eigenvalues must be (at least) doubly degenerate.

Some examples of supersymmetric systems are

(i) $Q_1=\partial$, $Q_2=\partial^*$, $H=\partial\partial^*+\partial^*\partial$ where ∂ is the outer derivative for a differentiable manifold and ∂^* is its dual. Examples of this kind have been used by Witten[1] and Gaume[2] to obtain new insights into Morse theory and the Atiyah-Singer index theorem.

(ii) $a_i = Q_i + iQ_{i+p}$ where $\{a_i, a_j^*\} = \delta_{ij}$ are the usual creation and destruction operators. These Q_i are often used to construct the finite-dimensional representations of the compact simple Lie groups[3], and then $H = \sum_i Q_i^2$ plays the role of a Casimir operator.

(iii) $Q = Q_1 \pm i Q_2$, where $\{Q_+, Q_-\} = \{Q_1^2 + Q_2^2\} = H$ and H is the Hamiltonian for a one-dimensional quantum-mechanical system[4] (This is the case that will be considered in the present lecture)

(iv) Q_α, $\alpha=1\ldots4$ where the Q are hermitian operators which transform according to the Majorana representation of the Lorentz group, and satisfy the relations $\{Q_\alpha, Q_\beta\} = (C\gamma_\mu)_{\alpha\beta}P^\mu$ $[P_\mu, Q_\alpha] = 0$ where C is the charge conjugation matrix and P_μ is the four-momentum. (This is the case of ordinary quantum field theory. and will be considered in the following three lectures).

(v) Case (iv), but with the restriction $[P_\mu, Q_\alpha] = 0$, or constancy of the Q_α, relaxed to $[P_\mu[P_\lambda, Q_\alpha]] = 0$, or linearity of the Q_α in x. This is the case of conformally invariant QFT.

(vi) Case (iv), but with the x-dependence of the Q_α left arbitrary (supergravity theory).

The literature on all these aspects of supersymmetry is enormous, but as we shall be interested only in its spontaneous breakdown we shall confine our attention to the quantum mechanical and the (non-gravitational) field theoretical cases (iii) and (ii) respectively, and concentrate only on the breaking in these cases. The quantum mechanical case will be considered in this first lecture and the field theoretical case in the following two lectures.

2.. Construction of Supersymmetric Quantum Mechanical Hamiltonians

Let us now consider case (iii) above — supersymmetric quantum mechanics in one dimension. Given any ordinary, one-dimensional, quantum-mechanical system

$$H_0 = \frac{p^2}{2} + V(x) \quad , \tag{1.4}$$

(where $V(x)$, which has to be bounded below for stability in any case, is normalized to be positive) a supersymmetric counterpart can be constructed as follows: Let U(X) denote a square root of 2V(X)

$$U^2(X) = 2V(X) , \tag{1.5}$$

and let

$$Q^\pm(x) = \left(U(x) \pm \partial/\partial x\right)\sigma^\pm , \tag{1.6}$$

where $\sigma^\pm$ are the usual Pauli matrices $(\sigma_1 \pm i\sigma_2)/2$, and $\partial/\partial x = ip$. Then the Hamiltonian

$$H = \frac{1}{2}\{Q^+, Q^-\} = \frac{1}{2}U(x)^2 - \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\sigma_3}{2}\frac{\partial U(x)}{\partial x} = H_0 + \frac{\sigma_3}{2}\frac{\partial}{\partial x}(V(x))^{1/2} , \tag{1.7}$$

is the supersymmetric counterpart of (1.4). Note that

$$[H, Q^\pm] = 0 \quad , \quad [H, \sigma_3] = 0 \quad , \quad \{Q^\pm, \sigma_3\} = 0 \quad , \tag{1.8}$$

so that σ_3 plays the role of σ. Note also that for $\sigma_3 = \pm 1$ H reduces to S^+S and SS^+ respectively where $S = U - \partial/\partial x$. It follows that H is non-negative and that if a state with zero eigenvalue of H exists, then it is unique and satisfies either

$$\sigma_3\psi_0 = \psi_0 \ , \ S\psi_0 = 0 \quad \text{or} \quad \sigma_3\psi_0 = -\psi_0 \ , \ S^{+}\psi_0 = 0 \tag{1.9}$$

All other energy eigenstates are doubly degenerate[5)6)]. From (1.7) it is clear that the rule for supersymmetrizing a given one dimensional quantum mechanical Hamiltonian $p^2/2 + V(x)$ is simply to add a term $\frac{\sigma_3}{2}\frac{\partial U(x)}{\partial x}$ where $U^2(X) = 2V(X)$. The only ambiguity that arises in this procedure is in the sign of the square root. If $V(X)$ is real analytic the ambiguity can be reduced to an overall sign of $U(X)$ by requiring that $U(X)$ also be real analytic. However, this is not a necessary condition, because there is no need for a potential to be analytic, differentiable or even continuous (piece-wise continuity is all that is required, as in the case of square-well potentials for example). In particular, if $V(x)$ has zeros, natural choices of $U(x)$ are obtained by choosing different signs on either side of the zeros. For example for the monomial potentials $2V(x) = \omega^2 x^{2n}$ the analytic square-root is ωx^n but other admissible square-roots are $U(x) = \pm\omega x^n \theta(x)$ where $\theta(x) = \pm 1$ for $x \gtrless 0$. Similarly, for $2V(x) = \prod(x-a_i)^2$ admissible square roots are $U(x) = \pm\prod(x-a_i)$ where the $\pm$ may be different in the different sections $x < a_1$, $a_1 < x < a_2$, $a_2 < x < a_3 \dots$. We shall return to the question of choosing $U(X)$ later as it plays an important role in symmetry breaking.

3. Supersymmetric Harmonic Oscillators

In order to obtain some intuitive feeling for the supersymmetric Hamiltonian (1.7) let us consider some examples. First, for the conventional harmonic oscillator potential $2V(x) = \omega^2 x^2$ with analytic square root $U(x) = \omega x$ the quantum mechanical counterpart is

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\sigma_3}{2}\omega \ , \tag{1.10}$$

and since the additional term $\omega\sigma_3/2$ is constant it is easy to see that it converts the conventional harmonic oscillator spectrum

$$E = \frac{\omega}{2}, \frac{3\omega}{2}, \frac{5\omega}{2}, \dots \ , \tag{1.11}$$

into the spectrum

$$\begin{aligned} E &= \quad , \omega , 2\omega , 3\omega , \dots && (\sigma_3 = 1) \\ &= 0 , \omega , 2\omega , 3\omega , \dots && (\sigma_3 = -1) \ . \end{aligned} \tag{1.12}$$

Thus, as expected, the ground state (described by the wave function $\exp(-\omega x^2/2)$) is unique and has zero energy, and all the other energy levels are doubly degenerate.

Consider next the (anharmonic) oscillator with potential $\omega^2 x^4/2$. An analytic square root is $U(x) = \omega x^2$ and then the supersymmetric counterpart is

$$H = \frac{p^2}{2} + \frac{\omega^2 x^4}{2} + \sigma_3 \omega x \ . \tag{1.13}$$

It is easy to see that if $\binom{f(x)}{g(x)}$ is an eigenstate of this Hamiltonian then $\binom{g(-x)}{f(-x)}$ is an eigenstate with the same eigenvalue, and that there is no state with eigenvalue zero, since the solutions $\exp(\pm\omega x^3/3)$ of the equations $S\psi_0 = 0$ and $S^{+}\psi_0 = 0$ are not square-integrable. Thus the energy spectrum must consist of a doubly degenerate set

$$E = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots \ (\sigma_3 = \pm 1) \qquad \text{where} \qquad \varepsilon_0 > 0 . \tag{1.14}$$

4. Symmetry Breaking.

The Hamiltonian (1.7) is formally supersymmetric in the sense that it commutes with the generators $Q^{\pm}$ of supersymmetry. However, following convention, the system will be said to be supersymmetric if, and only if, in addition, the ground state ψ_{ε_0} is supersymmetric i.e.

$$Q^{\pm}\psi_{\varepsilon_0}(x) = 0 \ , \tag{1.15}$$

and, as we have already seen, this will be true if, and only if, the ground state energy ε_0 is zero (and ψ_{ε_0} is unique). Thus the basic condition for supersymmetry breaking is $\mathcal{E}_0 > 0$. In analogy with the situation in field theory the supersymmetry in this case ($\mathcal{E}_0 > 0$) is *often said to be spontaneously broken*. However, it should be recalled that because quantum mechanics has only a finite number of degrees of freedom there is no spontaneous breakdown in the usual field-theoretic sense of the word. (The ground states $\psi^{\pm}_{\mathcal{E}_0}$ where $\sigma_3\psi^{\pm}_{\mathcal{E}_0} = \psi^{\pm}_{\mathcal{E}_0}$ can be transformed into one another by unitary transformations).

A simple practical criterium for supersymmetry breaking ($\mathcal{E}_0 > 0$) can be obtained from the condition, $S\psi_0 = 0$ or $S^{\dagger}\psi_0 = 0$ where $S = U - \partial/\partial x$, of Section 2, because, since $U - \partial/\partial x$ is a first-order differential operator these equations can be solved at once[5] to yield

$$\psi_{\mathcal{E}_0} = \text{const.} \times \exp\left(\pm \int_0^x U(y)\,dy\right), \tag{1.16}$$

and so the question reduces to the square-integrability of the expression in (1.16). This question in turn reduces to the question as to whether the leading term in $U(x)$ as $x \to \pm\infty$ is even or odd, since if $U(X)$ is odd/even, then $\int^x U(y)dy$ is even/odd and (1.16) is/is not square-integrable. Thus the criterium for symmetry breaking is simply that $U(x)$ be even for large x. Examples of this have already been seen in Section 3, where for the harmonic oscillator with $U(x) = \omega x$ the symmetry is unbroken and for the anharmonic oscillator with $U(x) = \omega x^2$ it is broken. More generally, for monomial potentials $2V(x) = \omega^2 x^{2n}$, $U(x) = \omega x^n$ it will be unbroken for odd n and broken for even n.

5. Non-Analytic Square Roots

The result just obtained for monomial potentials holds only for the analytic square-roots. If, for example, one chooses the alternative square-roots

$$U(x) = \omega x^n \theta(x) \quad \text{where} \quad \theta(x) = \pm 1 \text{ for } x \gtrless 0, \tag{1.17}$$

then it is easy to see that just the opposite situation obtains, since in this case $U(x)$ is odd/even according as n is even/odd. Note that the Hamiltonian (1.7) constructed with $U(x)$ in (1.17) is well-defined for $x \gtrless 0$ so long as $n \geq 1$, and is continuous and well-defined even at $x=0$ for $n \geq 2$. Thus the simplest case, namely the case of the non-analytic harmonic oscillator

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\sigma_3}{2}\omega\,\theta(x), \tag{1.18}$$

is the only case in which the potential for (1.17) is discontinuous, and even in this case it is no more discontinuous than a potential well. It is easy to verify that for (1.18) the lowest energy is strictly positive, and the lowest energy state is doubly degenerate.

In contrast, if one takes the non-analytic square-root $U(x) = \omega x^2 \theta(x)$ for the anharmonic oscillator $2V(x) = \omega^2 x^4$ then

$$H = \frac{p^2}{2} + \frac{\omega^2 x^4}{2} + \sigma_3\omega\,\theta(x)\,x, \tag{1.19}$$

and it is easy to verify that

$$\psi_0(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left(-\omega|x|^3/3\right), \tag{1.20}$$

is a (unique) ground state.

It is perhaps amusing to consider in the context of the supersymmetric harmonic oscillator (1.18) the (non-supersymmetric) modified harmonic oscillator

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \lambda\theta(x) , \qquad (1.21)$$

where λ is an arbitrary parameter. It is easily seen that if $\lambda = 2m$ where m is an integer, the eigenstates of this Hamiltonian are linear combinations of two ordinary harmonic oscillator states $|n)$ and $|n + 2m)$, but if $\lambda \neq 2m$ (as in (1.18)) then the eigenstates are not given by any finite combination of the $|n)$.

6. The Higgs Potential.

A final interesting example is provided by the standard Higgs potential

$$2V(x) = \lambda^2(x^2 - a^2)^2 . \qquad (1.22)$$

The analytic square root of (1.22) is evidently

$$U(x) = \pm\lambda(x^2 - a^2) , \qquad (1.23)$$

and, according to the criterium of Section 4, the supersymmetric counterpart of (1.22) constructed with (1.23) does <u>not</u> have a supersymmetric ground state since the leading term in (1.17) for large x is even. Thus the supersymmetry of the analytic supersymmetric counterpart of (1.22) is broken.

On the other hand, there are actually six natural square-roots of (1.22), namely (1.23) with the choices $\pm$ not universal, but chosen independently in the three sections

$$x \leq -a \qquad -a \leq x \leq a \qquad x \geq a , \qquad (1.24)$$

From (1.16), (1.23) and (1.24) it is easy to see that the supersymmetry will be unbroken if, and only if, one makes one of the two choices $(-, \pm, +)$ for the three regions in question. In that case

$$\frac{\partial U(x)}{\partial x} = \pm 2\lambda x \qquad \text{for } x \gtrless a \text{ and } -a , \qquad (1.25)$$

and the ground-state wave-function is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(\pm \frac{\lambda x}{3}(x^2 - 3a^2)\right) \qquad \text{for } x \gtrless -a, a , \qquad (1.26)$$

respectively. Note that in these cases the supersymmetry is preserved but the reflexion-symmetry $a \leftrightarrow -a$ is broken.

<u>A Remark on the Partition Function.</u> Since supersymmetry is characterized by the degeneracy of the energy levels, and its breaking by the non-existence of a ground state with zero energy, a useful function for describing it is the so-called partition function $\chi(T)$ defined as

$$\chi(T) = \sum_n m(n)\, e^{-E_n T} , \qquad (1.27)$$

where E_n are the energy levels and $m(n)$ their multiplicities. In particular, the limit $T \to \infty$ of $\chi(T)$ is sufficient to determine whether the supersymmetry is broken or not, since $\chi(T) \to 0,1$ in the respective cases. Note that $\chi(T)$ can be expressed as the analytic continuation from t to T = it of the Feynman-Kac integral $\int K(a,a,t)\,da$ where

$$K(a,b,t) = \langle b, e^{iHt} a\rangle = \int_a^b d[x] \exp i \int_0^t dt\, \mathcal{L}(x,\dot{x}) . \qquad (1.28)$$

Let us consider for example the case of the harmonic oscillator. It is well-known that in the non-supersymmetric case

$$K(a,b,t) = \left(\frac{\omega}{2\pi \sin\omega t}\right)^{1/2} \exp\left\{\frac{\omega}{2\sin\omega t}\left[(a^2+b^2)\cos\omega t - 2ab\right]\right\} , \qquad (1.29)$$

so the partition function is

$$\chi(T) = \frac{1}{\sinh\omega T} = e^{-\frac{\omega T}{2}}\left(1 + e^{-\omega T} + e^{-2\omega T} + \dots\right), \qquad (1.30)$$

reflecting the fact that the energy levels are $E_n = (n+\frac{1}{2})\omega$ and that they all have multiplicity one. As already seen in Section 1.2 the (analytic) supersymmetric analogue is obtained by simply adding a term $\sigma_3\omega/2$ to the Hamiltonian, and from (1.10) one sees that this makes the simple change

$$\chi(T) \to \chi_{ss}(T) = \frac{\mathrm{tr}_R\, e^{\sigma_3\omega T/2}}{\sinh\omega T} = \frac{\cosh\omega T}{\sinh\omega T} = \left(1 + 2e^{-\omega T} + 2e^{-2\omega T} + \dots\right), \qquad (1.31)$$

in the partition function. As expected, $\chi_{ss}(\infty) = 1$ since the supersymmetry is unbroken, and the non-zero eigenvalues of H are multiples of ω and are doubly degenerate.

It might be worth remarking that if one changes the trace in (1.31) to the supersymmetric trace

$$\mathrm{str}_R\, e^{\sigma_3\omega T/2} \to \mathrm{str}\, e^{\sigma_3\omega T/2} = \mathrm{tr}(-1)^F e^{\sigma_3\omega T/2} = \mathrm{tr}\,\sigma_3 e^{\sigma_3\omega T/2}, \qquad (1.32)$$

where F is the 'fermion' number, which in this case is just ± 1 for $\sigma_3 = \pm 1$ one finds that

$$s\chi_{ss} \equiv \frac{\mathrm{str}\, e^{\sigma_3\omega T/2}}{\sinh\omega T} = 1, \qquad (1.33)$$

for all T. This is actually the quantum-mechanical-harmonic-oscillator analogue of the Nicolai[7] result

$$\frac{1}{N}\int d(A,\chi)\, e^{-\int \mathcal{L}_{ss}(A,\chi)\,dx} = 1, \qquad (1.34)$$

in unbroken supersymmetric field theory. This result states that the Schwinger functional for zero external current (J=0) is independent of the parameters in the supersymmetric Lagrangian $\mathcal{L}_{ss}(A,\chi)$. It implies that in such a theory the vacuum graphs must vanish, a result that had already been observed in perturbation theory[8].

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LECTURE 2. SUPERSYMMETRY IN QUANTUM FIELD THEORY.

2.1. Superfields and Supertranslations

In this and the following two lectures we shall consider the spontaneous breakdown of supersymmetry in quantum field theory (QFT). But to do this it is necessary to describe supersymmetry in QFT, and that will be the purpose of the present lecture[1]. As already mentioned in the first lecture the basic commutation relations to be satisfied in the QFT case are

$$\{ Q_\alpha , Q_\beta \} = i(C\gamma^\mu)_{\alpha\beta} P_\mu \quad , \tag{2.1}$$

where the Q_α are the supersymmetric generators, C is the charge conjugation matrix, P_μ is the four-momentum, and it is understood that

$$[P_\mu, Q_\alpha] = 0 , \qquad [L_{\mu\nu}, Q_\alpha] = M^{\alpha\beta}_{\mu\nu} Q_\beta \quad , \tag{2.2}$$

where $M^{\alpha\beta}_{\mu\nu}$ are the generators of the Lorentz group in the Majorana (real Dirac) representation. The problem is how to implement the Q_α on quantized fields.

The solution to this problem is most neatly expressed by introducing the ideas of superspace and superfields[2]. Superspace is obtained by adding to the usual coordinates x_μ for a four-dimensional space with Minkowski metric, four anti-commuting coordinates θ_α

$$\{\theta_\alpha, \theta_\beta\} = 0 \quad , \tag{2.3}$$

for a 4-dimensional space with a symplectic metric. Superfields $\Phi(x,\theta)$ are then fields over the resulting eight dimensional space. They satisfy the usual Lorentz transformation law

$$\left(U(\Lambda,a)\Phi\right)(x,\theta) = D(\Lambda)\,\Phi(\Lambda^{-1}(x-a), M^{-1}\theta) , \tag{2.4}$$

where $D(\Lambda)$ is some finite-dimensional representation of Λ (usually chosen to be trivial, $D(\Lambda) = 1$)) and $M(\Lambda)$ is the Majorana representation of Λ . The new feature is that they also satisfy the 'supertranslation' law

$$\left(U(\varepsilon)\Phi\right)(x_\mu,\theta) = \Phi\left(x_\mu + i\,\bar{\theta}\gamma_\mu\varepsilon \;,\; \theta+\varepsilon \right), \tag{2.5}$$

where ε_α are a second set of anti-commuting coordinates (analogues of a_μ for the Poincaré group) which also anti-commute with the θ's. Thus (2.5) is essentially a translation in the θ-part of the superspace, but it induces also a formal translation $x_\mu \to x_\mu + i\bar{\theta}\gamma_\mu\varepsilon$ in the Minkowski part. (Of course, one may take more than one set of θ's, i.e. θ^i_α , i=1...n, in which case the supersymmetry is called n-supersymmetry, but we shall consider only n=1 in these lectures).

By taking the infinitesimal (linear in ε) version of the supertranslations (2.5) one sees that

$$\delta\Phi = U(\varepsilon)\Phi - \Phi = \varepsilon_\alpha Q_\alpha \Phi \quad , \tag{2.6}$$

where

$$Q_\alpha = i\frac{\partial}{\partial\theta_\alpha} + (\bar{\theta}\gamma)_\alpha \quad . \tag{2.7}$$

It is then easy to verify that the Q_α satisfy the relations (2.1) and (2.2), so these relations have been implemented at the level of the superfields.

To implement them at the level of ordinary fields, one notes that since the θ_α anti-commute, the superfields have a finite expansion in terms of ordinary fields

$$\Phi(x,\theta) = A + \bar{\theta}\cdot\eta + \bar{\theta}\,(\gamma^\mu A_\mu + F_1 + \gamma_5 F_2)\theta + (\bar{\theta}\cdot\theta)(\bar{\theta}\cdot\lambda) + (\bar{\theta}\theta)^2 D , \tag{2.8}$$

where the ordinary component fields not only have the usual Lorentz properties as indicated (Φ scalar $\rightarrow$ A, F, D scalars, F_2 pseudo-scalar, A_μ vector, η, λ Majorana spinors) but also have the conventional commutation and anti-commutation relations when the superfields have them (e.g. the commutation of $\Phi(x,\theta)$ and $\Phi(y,\theta)$ implies the commutation of A(x) and A(y) and the anti-commutation of $\eta(x)$ and $\eta(y)$). By applying (2.7) to (2.8) one finally obtains the operation of the Q_α on the ordinary fields as

$$(\varepsilon\cdot Q)\,A(x) = \bar{\varepsilon}\gamma_5\eta(x)$$

$$(\varepsilon\cdot Q)\,\eta(x) = \left\{\not{\partial}A(x) + \underset{\sim}{\gamma}\,\underset{\sim}{F}(x) + \gamma_5\not{\partial}C(x)\right\}\varepsilon$$

$$(\varepsilon\cdot Q)\,\underset{\sim}{F}(x) = \bar{\varepsilon}\,\underset{\sim}{\gamma}\left(\lambda(x) + \not{\partial}\eta(x)\right) \qquad (2.9)$$

$$(\varepsilon\cdot Q)\,A_\mu(x) = \bar{\varepsilon}\left(\gamma_\mu\lambda(x) + \partial_\mu\eta(x)\right)$$

$$(\varepsilon\cdot Q)\,\lambda(x) = \left\{\gamma_5 D(x) + \vec{\sigma}\cdot\vec{B}(x)\right\}\varepsilon$$

$$(\varepsilon\cdot Q)\,D(x) = \bar{\varepsilon}\gamma_5\not{\partial}\lambda(x)$$

$$\underset{\sim}{F}\cdot\underset{\sim}{\gamma} = F_1 + \gamma_5 F_2 \qquad \vec{\sigma}\cdot\vec{B} = \sigma_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)\,.$$

Thus the superfield and supertranslations may be thought of as just a convenient way of summarizing (2.9). Note that the super multiplit (A, F, D, A_μ, η, λ) contains component fields of different spin, including both fermions and bosons, but that (modulo spontaneous supersymmetry breaking, all the component fields must have the same mass since $P_\mu P^\mu$ commutes with the Q_α. Note also that the two parts, $\partial/\partial\theta_\alpha$ and $(\bar{\theta}\gamma)_\alpha$ of the Q_α may be thought of as step-up and step-down operators respectively, which are similar to those in the representations of SU(3) or indeed of any compact simple group, except that they are both accompanied by a spin change $\Delta s = \pm\frac{1}{2}$ and that the step-down operation is accompanied by a Euclidean-space derivative.

2.2 Reduction of Superfields

As it stands the field $\Phi(x,\theta)$ does not carry an irreducible representation of the supertranslations. Indeed there are two ways that it can be reduced, corresponding roughly to the reduction of a Dirac spinor to two Majorana spinors and to two chiral (Weyl) spinors respectively.

The first (Majorana) reduction is simply to demand that $\Phi(x,\theta)$ be real, and this puts the following reality conditions on the component fields

$$\Phi^*(x,\theta) = \bar{\Phi}(x,\theta) \quad \Leftrightarrow \quad (\,A, F_1, F_2, D \text{ real: } \eta, \lambda \text{ Majorana}). \qquad (2.10)$$

Such a superfield is called a real vector superfield because the highest spin it contains is a vector.

The second (chiral) reduction is more complicated. First, one notices that

$$\{Q_\alpha, D_\beta\} = 0 \qquad \text{where} \qquad D_\beta = i\frac{\partial}{\partial\theta_\beta} - (\bar{\theta}\gamma)_\beta\,, \qquad (2.11)$$

that is, if one introduces the operators D_β which are the differences rather than the sums of the step-up step-down operators then they commute with the Q_β. It follows that the conditions

$$D_\alpha\Phi = 0 \qquad \text{or} \qquad D^*_\alpha\Phi = 0\,, \qquad (2.12)$$

on a superfield are supersymmetric invariant. Applied to a real superfield the conditions (2.12) give nothing (they kill it) but applied to a complex superfield, they reduce the independence of the component fields as follows:

$$D \rightarrow \Box A \qquad \lambda \rightarrow \not{\partial}\eta \qquad A_\mu \rightarrow \partial_\mu A\,. \qquad (2.13)$$

Superfields satisfying (2.12) are denoted $\Phi^\pm$ and are called chiral scalar superfields (because the vector has been reduced to a scalar and the fermion η to its chiral components).

The conditions (2.12) can also be expressed in superfield language in the following way. Let $\theta, \theta^* = (\theta_a, \theta_{\dot{a}})$, $a, \dot{a} = 1,2$ be the chiral (Weyl) components of θ_α, $\alpha = 1\ldots4$ and

$$\Psi(x,\theta) = A + \bar{\theta}\cdot\chi + F(\bar{\theta}\cdot\theta), \qquad (2.14)$$

and its complex conjugate $\Psi^*(x,\theta^*)$ be superfields constructed with the two-component θ (and θ^*) alone. Then

$$\Phi^+(x,\theta) = \Psi(x_\mu + i\bar{\theta}\gamma_\mu\theta, \theta), \qquad (2.15)$$

and similarly for the complex conjugate. In other words the dependent multiplets $(A, \eta, F, \partial_\mu A, \not\partial\eta, \Box A)$ are just the simpler multiplets (A, χ, F) boosted by letting $x \to x + i\bar{\theta}\gamma\theta$. Note that for the chiral multiplets $\Psi(x,\theta)$ one can introduce the chiral transformations

$$\Psi(x,\theta) \to e^{in\alpha}\Psi(x, e^{i\alpha}\theta), \qquad (2.16)$$

where n is an arbitrary integer. Symmetry with respect to this transformation is called R-symmetry, and the integers n are called R-charges.

We shall see later that the chiral superfields play the role of matter fields (spin 0 and $\frac{1}{2}$ fields) and we should like the real vector fields Φ to play the role of gauge-fields. For this purpose, however, one has to generalize to superfields the concept of gauge transformations

$$A_\mu \to G^{-1}A_\mu G + G^{-1}\partial_\mu G \qquad \text{where} \qquad G = \exp i\Lambda(x), \qquad (2.17)$$

where A_μ and Λ lie in the Lie algebra of a compact Lie group. To find the generalization is not so easy but, once found, it turns out to be even simpler than (2.17), as follows. Consider r real vector fields Φ_a assigned to the adjoint representation of a simple Lie group or the 1-dimensional representation of an abelian one and let $\Phi^\pm_n$ be n chiral scalar superfields assigned to any representation of the Lie group (and its conjugate). Then the required generalization of the gauge-transformation (2.17) is

$$\exp\Phi \to \exp i\Phi^+ \exp\Phi \exp(-i\Phi^-) \qquad \text{where} \quad \Phi = \Phi_a t_a \qquad (2.18)$$

and t_a are the generators of the Φ representation.

Since $\Phi = (A, \eta, F, A_\mu, \lambda, D)$ and $\Phi^+ = (B, \chi, F)$ it is intuitively evident that a gauge can be chosen so that the fields (A, η, F) of are cancelled by the corresponding fields in $\Phi^\pm$. Such a gauge is called the Wess-Zumino gauge[3], and since in this gauge Φ consists only of (A_μ, λ, D) it is obviously the most convenient gauge for the component field formulation. The only residual gauge freedom in the Wess-Zumino gauge is the usual one (2.16) where $\Lambda = B + B^*$, and where λ and D are gauge-invariant.

2.3 Construction of Lagrangians

The construction of supersymmetric Lagrangians is based on the following two simple principles

(i) Products of superfields, such as $\Phi^+\Phi^-$, Φ^2, $(\Phi^+)^3$ are again superfields.

(ii) The variation of the "highest" components $(\Phi^+)_h = F$, $(\Phi)_h = D$, $(\Phi^2)_h$, $(\Phi^{+3})_h$ etc. of superfields are total derivatives.

It follows that the supersymmetric variation of a quantity such as

$$\mathcal{L} = a(\Phi^+\Phi^-)_h + b(\Phi)_h + c(\Phi^2)_h + d(\Phi^{+4} + \Phi^{-4})_h, \qquad (2.19)$$

consists only of a derivative. Hence, for conventional boundary conditions the variation of $\int d^4x\, \mathcal{L}_{ss}$ is zero.

Before going on to discuss the particular Lagrangians used for the matter and gauge superfields of Section 2, it might be worthwhile to illustrate first the importance of the boundary conditions by noting that for finite temperature the argument fails and the supersymmetry[4] becomes broken. The point is that for finite temperature the time-integral in $\int \mathcal{L}\, d^4x$ is finite, $t_0 \le t \le t_0 + 2\pi/T$, where T is the termperature). At the same time, because the Lagrangian is bosonic its variation must be fermionic,

$$\delta \mathcal{L}(x) = \partial_\mu \mathcal{F}_\mu(x) , \tag{2.20}$$

where $\mathcal{F}_\mu(x)$ is fermionic, and since for fermions the boundary conditions are <u>anti</u>-periodic, one obtains

$$\delta \int_{t_0}^{t_0+\frac{2\pi}{T}} dt \int d^3x\, \mathcal{L}(x) = Q(t_0+\tfrac{2\pi}{T}) + Q(t_0) = 2Q(t_0) \text{ where } Q(t_0) = \int d^3x\, \mathcal{F}_0(x,t_0), \tag{2.21}$$

and $Q(t_0)$ does not vanish since otherwise the charge would be trivial. Thus, in contrast to the usual variation $Q(t_1) - Q(t_2)$ which vanishes either because $t_1, t_2 \to \pm\infty$ and $Q(\pm\infty)$ vanishes, or because of periodic boundary conditions $Q(t_1) = Q(t_2) (\neq 0)$, the variation in (2.21) does not vanish and there is a spontaneous breakdown of the supersymmetry. Physically this may be understood from the fact that, in spite of the supersymmetry, the 'filling-up' of boson and fermion states is not the same on account of the different statistics, so that at any temperature $n_B \neq n_F$. All that the supersymmetry implies is that the <u>change</u> in these numbers in a single process be equally probable, $\Delta n_B = \Delta n_F$.

2.4 Supersymmetric Lagrangians for Matter Superfields

For the matter superfields it turns out that the most general renormalizable Lagrangian for the chiral scalar superfields $\Phi^\pm$ is

$$\mathcal{L}_{SS} = \int d^4x \left\{ (\Phi^+\Phi^-)_h + f(\Phi^+ + \Phi^-)_h + m(\Phi^{+2} + \Phi^{-2})_h + g(\Phi^{+3} + \Phi^{-3})_h \right\}. \tag{2.22}$$

On expansion in terms of the conventional fields (A, χ, F) in the supermultiplets this turns out to be

$$\mathcal{L}_{SS} = \int d^4x \left\{ \tfrac{1}{2}(\bar\chi \partial\!\!\!/ \chi + (\partial A)^2 + F^2) + fF + m(\bar\psi\psi + AF + \bar A \bar F) + g(\bar\psi\psi A + FA^2 + h.c.) \right. . \tag{2.23}$$

Since the field f has no kinetic term it can be eliminated using the Euler-Lagrangian equations

$$\frac{\delta \mathcal{L}}{\delta F} = F + f + mA + gA^2 = 0, \tag{2.24}$$

and then one obtains

$$\mathcal{L}_{SS} = \int d^4x \left\{ \bar\psi(\partial\!\!\!/ + m + gA)\psi + \tfrac{1}{2}|F|^2 \right\} \tag{2.25}$$

where

$$\bar\psi \Lambda \psi = \bar\psi (A_1 + i A_2 \gamma_5) \psi , \qquad (-F) = f + mA + gA^2 . \tag{2.26}$$

In other words one obtains the usual renormalizable Lagrangian for a Majorana field coupled to a scalar and pseudo-scalar field, but with the masses and coupling constants strongly correlated, indeed reduced to the three parameters f, m, g. The generalization to any number of superfields Φ_a is evidently

$$\mathcal{L}_{SS} = \int d^4x \left\{ \bar\psi_a \partial\!\!\!/ \psi_a + \bar\psi_a m_{ab} \psi_b + g_{abc} \bar\psi_a A_b \psi_c + \tfrac{1}{2} F_a \bar F_a \right\}, \tag{2.27}$$

where

$$- F_a = f_a + m_{ab} A_b + g_{abc} A_b A_c , \tag{2.28}$$

and the couplings m_{ab} and g_{abc} are totally symmetric.

5. Lagrangians for Gauge-Superfields.

In order to construct the Lagrangian for gauge superfields we first recall the construction for ordinary gauge-fields, namely, if $\mathcal{L}(\phi, \partial_\mu \phi)$ is a non-gauge Lagrangian then the gauge-equivalent is

$$\mathcal{L}(\phi, \mathcal{D}_\mu \phi) + \frac{1}{4} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} , \tag{2.29}$$

where D_μ is the covariant derivative $\partial_\mu + e A_\mu$, where $A_\mu = A_\mu^k \sigma_k$ are the gauge-potentials, and $F_{\mu\nu}$ are the gauge-fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu] . \tag{2.30}$$

The procedure for superfields is quite analogous, but in order to describe it, one needs to have the supersymmetric analogues of the gauge-fields $F_{\mu\nu}$. The form of these is a little unexpected, namely,

$$W_\alpha = D^*_\beta D^{*\beta} (e^{-e\Phi} D_\alpha e^{e\Phi}) , \tag{2.31}$$

where D_α and D^*_α are the differential operators defined in Section 2 and which anti-commute with the supersymmetric generators Q_α. On expansion, the W_α take the form

$$W_\alpha = (\lambda_\alpha, F_{\mu\nu}, D, \dots) , \tag{2.32}$$

where the fields not written vanish in the Wess-Zumino (WZ) gauge. The procedure for gauging the supersymmetric matter-Lagrangian (2.27) is then quite analogous to (2.29), namely,

$$\mathcal{L}_{ss}^{matter} \Rightarrow \mathcal{L}_{ss} = \int d^4x \{ \bar{\Phi}^+ e^{e\Phi} \Phi^- + \frac{1}{4} \mathrm{tr}\, W_\alpha W^\alpha + d\, \mathrm{tr}\, D \} , \tag{2.32}$$

where the second term, like $\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$ in (2.29), is the supersymmetric Lagrangian for the super-gauge-field alone. On expansion in terms of the conventional fields (in the WZ - gauge) this term yields

$$\mathrm{tr}\, W_\alpha W^\alpha = \mathrm{tr}(\bar{\lambda} \not{D} \lambda + F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^2) , \tag{2.33}$$

where D^2 plays no role (the field equations for (2.33) imply $D = 0$) and the first two terms are just the conventional terms for a gauge-field interacting with a Majorana field in the adjoint representation. Thus, such a conventional (F, λ) system is automatically supersymmetric!

The matter-gauge-field interaction that results from (2.32) (in the WZ - gauge) is just

$$\begin{aligned} &\bar{\chi}(\not{D} + m + gA)\chi + \frac{1}{2}|D_\mu A|^2 + \frac{1}{2}|F|^2 \\ &+ e\bar{\chi} A \lambda + e A^+ D A + d\, \mathrm{tr}\, D . \end{aligned} \tag{2.34}$$

Thus it is just the usual term that one obtains from the substitution $\partial \to D$, together with a term $e\bar{\chi} A \lambda$ which links the two fermion fields, and two D-field terms. Since there are no kinetic terms for the D fields the latter can be eliminated by using the Euler-Lagrange equations, and on eliminating them, and introducing the Dirac fermion fields

$$\psi = \chi + i\lambda , \tag{2.35}$$

the expression (2.34) reduces to

$$\bar{\chi}(\not{D} + m + gA)\chi + \frac{1}{2}|DA|^2 + \frac{1}{2}|F|^2 + \frac{1}{2}|G|^2 , \tag{2.36}$$

where

$$(-G) = (d + eA^+A)^2 + e_c^2 (A^+ \sigma_k A)^2 , \tag{2.37}$$

where σ_k are the generators and e_c is/(are) the coupling constant (s) for the semi-simple part of the group.

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3) Wess, J. and Zumino, B., Nucl. Phys. B78 (1974) 1.

4) Girardello, L., Grisaru, Salomenson, P., Nucl. Phys. B178 (1981) 513. (For a counter-interpretation see van Hove, L., Nucl. Phys. B207 (1982) 15).

LECTURE 3. SPONTANEOUS SYMMETRY BREAKING IN SUPERSYMMETRIC FIELD THEORY

1. Variety of Symmetry Breakdown

Having set up the standard supersymmetric Lagrangians for matter and gauge fields we can now turn to the real purpose of these lectures, namely the spontaneous breakdown of these Lagrangians. There are actually two distinct kinds of spontaneous symmetry breaking namely (a) the spontaneous breakdown of an internal symmetry in the presence of supersymmetry and (b) the spontaneous breakdown of supersymmetry itself. Let us consider the general features of each kind in turn.

(a) Spontaneous breakdown of internal symmetry. Such a spontaneous breakdown is characterized by

$$\langle A \rangle \neq 0 \qquad\qquad \langle F \rangle = \langle D \rangle = 0 \, , \tag{3.1}$$

where A is the lowest and F,D the highest scalar fields in a supermultiplet. The reason that the supersymmetry is not broken in this case is that the fields A are themselves trivial superfields $\langle A \rangle = \Phi(0) = \Phi(x,\theta,\bar{\theta})$. By $\langle A \rangle$ in (3.1) is meant, of course, the value of A at the potential minimum in a classical theory or the vacuum value in a quantized theory. A breakdown of the internal symmetry is very easy to achieve because the matter-field potential is $|\;|^2/2$, and so any non-zero solution of the equation

$$F = f + mA + gA^2 = 0 \, , \tag{3.2}$$

will break the internal symmetry. Two novel features of internal symmetry breaking in the supersymmetric case are:

(i) The existence of Goldstone multiplets (including Goldstone fermions). This is because the supersymmetry is unbroken and therefore each Goldstone boson must be accompanied by a whole multiplet.

(ii) The existence of a Higgs mechanism for fermions. That is, the merging of scalar and gauge-fields in the usual Higgs mechanism is accompanied by a merging of matter-field and gauge field fermions.

The way in which this can happen has been foreshadowed by the merging of χ and λ into the Dirac field ψ, in (2.35).

(b) Spontaneous breakdown of supersymmetry itself. The spontaneous breakdown of supersymmetry itself is characterized by the opposite situation to (3.1), namely

$$\langle A \rangle = 0 \qquad \langle F \rangle \text{ or } \langle D \rangle \neq 0. \tag{3.3}$$

One sees that (3.3) implies supersymmetry breaking because

$$F = \{Q_\alpha, \chi\} \qquad D = \{Q_\alpha, \lambda\} \tag{3.4}$$

from (2.9) and hence (3.3) implies that $Q_\alpha|0\rangle$ cannot be zero. In contrast to the breakdown of internal symmetry, (3.3) is very difficult to achieve because if there is both a supersymmetric vacuum $|s\rangle$ and a non-supersymmetric vacuum $|ns\rangle$, then

$$H = \sum_\alpha Q_\alpha^2 \quad \Rightarrow \quad \langle s|H|s\rangle = 0 \ , \quad \langle ns|H|ns\rangle > 0 \tag{3.5}$$

so the supersymmetric vacuum is always the one with lower energy. Hence the only way in which supersymmetry can be broken is when no supersymmetric vacuum exists. In the case that a breakdown of supersymmetry does occur, there are again some novel features, namely,

(i) The existence of Goldstone fermions. However, in this case the Goldstone fermions exist in their own right because Goldstone fields are those in the directions of the variations of the vacuum state, and since

$$\delta F = \varepsilon \cdot \chi \qquad \delta D = \varepsilon \cdot \lambda \ , \tag{3.6}$$

one sees that in the supersymmetric case these are just the directions of χ and λ for $\langle F \rangle \neq 0$ and $\langle D \rangle \neq 0$ respectively.

(ii) The existence of anomalies similar to the Adler anomalies for axial vector currents[1]. More precisely, if Γ is the effective potential, and Q_α the supersymmetric generators, then, in contrast to what one would expect for a spontaneously broken symmetry, $Q_\alpha\Gamma$ is not zero.

2. Spontaneous Breaking of Internal Symmetry for Matter-Superfields.

Let us consider in this section the case where there are no gauge-fields i.e. the Lagrangian is that of (2.27) for matter fields alone, with potential

$$V = \tfrac{1}{2}|F|^2, \quad \text{where} \quad F_\alpha = f_\alpha + m_{\alpha\beta}A_\beta + g_{\alpha\beta\gamma}A_\beta A_\gamma \ . \tag{3.7}$$

Let us first consider the spontaneous breakdown of internal symmetry, and in particular consider two examples, in one of which the breaking is optional, in the other mandatory.

Example (1): Φ_a belongs to the adjoint representation of SU(n). Then

$$f_a = 0 \qquad m_{ab} = m\delta_{ab} \qquad \text{and} \qquad g_{abc} = g\, d_{abc} \ , \tag{3.8}$$

where d_{abc} is the totally symmetric invariant tensor used to construct the cubic Casimir invariant, equation (3.7) reduces to

$$F_a = m A_a + g\, d_{abc} A_b A_c \ , \tag{3.9}$$

and there are two solutions of $F_a=0$ namely

$$A_a=0 \qquad \text{and} \qquad A_a = \frac{m}{g}\,\mathrm{diag}(1,1,1,\ldots 1,\ 1-n) \qquad (3.10)$$

The first solution leaves the internal symmetry unbroken, and the second solution breaks it down to U(n-1). Neither solution is preferred at this level since V=0 for each one and thus the breakdown is optional. As a matter of fact, radiative corrections do not change this situation[2], but gravitational corrections may do so[3].

Example (2): To remove the degeneracy found in Example (1), and actually present for all irreducible representations, one uses reducible representations. The simplest example is obtained by using two representations of a group, one of which (S) is a singlet, and the other (A) any non-trivial (real) representation. Then, on choosing

$$V_{SS} = f(\Phi_S^+ + \Phi_S^-) + m(\Phi_A^{+2} + \Phi_A^{-2}) + g(\Phi_S^+\Phi_A^{+2} + \Phi_S^-\Phi_A^{-2}), \qquad (3.11)$$

one has

$$2V = |F_S|^2 + |F|^2, \qquad F_S = f + gA^2, \quad F_A = mA + gSA, \qquad (3.12)$$

and for $f/g<0$ the potential minimum is at

$$V=F=0 \quad\Rightarrow\quad S = -\frac{m}{g} \qquad |A|^2 = -\frac{f}{g}\ (>0), \qquad (3.13)$$

and since $|A| \neq 0$ the internal symmetry is necessarily broken.

At first sight the choice (3.11) would seem to be an ad hoc one, which would change after radiative corrections, because although a term of the form of A^3 in V could be avoided by using a representation with no cubic invariant, there is no way in which terms such as aS and bS^2 could be avoided by means of arguments based on internal symmetry, and in the presence of such terms

there is an unbroken solution of (3.12), namely,

$$S = -\frac{f}{a} \qquad \text{or} \qquad S^2 = -\frac{b}{f}, \qquad A=0. \qquad (3.14)$$

However, it turns out that terms such as aS, bS^2 (and A^3) can be eliminated by means of another symmetry, namely, the R-symmetry introduced in Section 2.2. By inspection of the Lagrangian (2.22) one sees that if Φ_a are different multiplets, with R-charges then one has the following selection rules:

$$\begin{array}{lll} f_a = 0 & \text{unless} & N_a = 2 \\ m_{ab} = 0 & \text{"} & N_a + N_b = 2 \\ g_{abc} = 0 & \text{"} & N_a + N_b + N_c = 2\,. \end{array} \qquad (3.15)$$

Hence if one chooses $N_S=2$ and $N_A=-1$ as the R-charges of the fields S and A of the model the Lagrangian can only take the form shown in (3.11). Thus the form (3.11) follows from symmetry principles and is stable with respect to radiative corrections.

3. Spontaneous Breakdown of Matter Supersymmetry

We have seen that the condition for a spontaneous breakdown of the supersymmetry itself is $F_a \neq 0$ where

$$F_a = \lambda_a + m_{ab}A_b + g_{abc}A_bA_c\,. \qquad (3.16)$$

However, since every quadratic equation (for complex variables) has a ro it is evident that for a single matter-superfield Φ there always exists an A such that F=0. Thus for a single matter-superfield there can be no breakdown of supersymmetry. The question, therefore, is whether F_a in (3.17) can be strictly non-zero for $a=1\ldots n$, $n\geq 2$. It seems that for n=2 the answer is again negative (although I know of no formal proof). But for $n\geq 3$ the answer is positive. Thus for $n\geq 3$ there can be a spontaneous breakdown of supersymmetry. In this lecture we shall describe the simplest example, which is for n=3. The general conditions for which F_a in (3.16) cannot be zero (for arbitrary $n\geq 3$ and complex

A_α) are not known and it might be an interesting problem to try to find them.

In any case, the example for n=3 is constructed as follows[4]: Let Φ_0, Φ_1, Φ_2 be three superfields, with R-charges

$$N_0 = N_2 = 1, \; N_1 = 0 , \tag{3.17}$$

respectively, and let the Lagrangian be R-invariant. Let us also suppose that the Lagrangian is invariant with respect to the following reflexion symmetry:

$$\Phi_0 \leftrightarrow \Phi_0 , \qquad \Phi_i \leftrightarrow -\Phi_i , \qquad i=1,2 . \tag{3.18}$$

It is easy to see that the most general renormalizable potential which is invariant with respect to these symmetries is unique and that it takes the form

$$V_{ss} = f(\Phi_0^{+} + \Phi_0^{-}) + m(\Phi_1^{+}\Phi_2^{+} + \Phi_1^{-}\Phi_2^{-}) + g(\Phi_0^{+}\Phi_1^{+2} + \Phi_0^{-}\Phi_1^{-2}) . \tag{3.19}$$

One then has

$$F_0 = f + gA_1^2 \qquad F_1 = mA_2 + gA_0A_1 \qquad F_2 = mA_2 . \tag{3.20}$$

From (3.20) one easily sees that $|F|^2 \geq f^2$ and that the lower bound is attained for

$$A_1 = A_2 = 0 , \qquad A_0 \ \text{arbitrary}. \tag{3.21}$$

The arbitrariness of at least one of the fields at the potential minimum seems to be a general feature of the spontaneous breakdown of supersymmetry for matter-superfields, (but again I do not know of any general proof, and it is found that the arbitrariness does not survive the radiative corrections).

It is easy to see that for $F \neq 0$ the fields χ_0 and A_0 are massless i.e. are the Goldstone fields. On computing the masses of the other fields at the potential minimum one finds that

$$m^2(A_2) = m^2(\chi_2) = m^2 \qquad \text{but} \quad \begin{aligned} m^2(\mathrm{Re}\,A_1) &= m^2 + 2fg \\ m^2(\mathrm{Im}\,A_1) &= m^2 - 2fg , \\ m^2(\chi_1) &= m^2 \end{aligned} \tag{3.22}$$

Thus the masses of the Φ_1 supermultiplet are split. It is, perhaps, worth noting that the mass-splitting of a supermultiplet is the best guarantee that the supersymmetry really is broken. Note, that even after breaking, the sum-rule

$$m^2(\mathrm{Re}\,A_1) + m^2(\mathrm{Im}\,A_1) + m^2(\chi_1^{+}) + m^2(\chi_1^{-}) = 4m^2 , \tag{3.23}$$

is preserved (a result that is also true for the soft explicit breaking that is obtained[5] simply by adding to the Lagrangian a term of the form $\int\lambda\lambda$. As mentioned earlier, the Ward identities i.e.

$$Q_\alpha \int d^4x \mathcal{L} = 0 , \tag{3.24}$$

are not preserved by the radiative corrections. In fact, for the one-loop effective action $\Gamma^{(1)}$ one obtains[1]

$$Q_\alpha \Gamma^{(1)} = f \int d^4x \left(A_0(x)\chi_0(x)\right) , \tag{3.25}$$

where A_0, χ_0 are the Goldstone fields. Note that the anomaly vanishes when $f = 0$.

4. Spontaneous Symmetry Breaking for Gauge Superfields

In order to consider spontaneous breakdown in the presence of gauge superfields let us recall that the Lagrangian that describes the interaction of matter with gauge-fields takes the form

$$\mathcal{L}_{ss}^{gauge}(x) = \Phi^{+}(\exp e\Phi)\Phi^{-} + \frac{1}{4}\mathrm{tr}(W_\alpha W^\alpha + \alpha D) + \\ + \{ f\Phi + m\Phi^2 + g\Phi^3 + \bar\chi(\not\partial + m + gA)\chi \} ? \tag{3.26}$$

where the last bracket is a short-hand notation for the self-interaction of the matter-fields, and that when it is ex panded in terms of conventional fields $\mathcal{L}$ becomes

$$\mathcal{L}^{gauge}_{ss}(x) = \mathrm{tr}\,(\bar{\lambda}\not{\partial}\lambda + F_{\mu\nu}F^{\mu\nu} + dD) \quad + e\bar{\lambda}A\chi + \bar{\chi}(\not{\partial} + m + gA)\chi + \tfrac{1}{2}(|F|^2 + |D|^2), \quad (3.27)$$

where F,λ are in the adjoint representation of the gauge-group, the matter-fields χ, A are in an arbitrary representation and

$$|F(A)|^2 = |f + mA + gA^2|^2 \qquad |D(A)|^2 = (d + e|A|^2)^2 + e_0(A^{+}\sigma A)^2. \quad (3.28)$$

The spontaneous breakdown is governed by $|F(A)|^2$ and $|D(A)|^2$ since these are the potentials, but since the case of F(A) alone has previously been discussed we shall assume for simplicity that F(A) plays no role (in particular that F(0)=0) and concentrate on D(A)[6]. Once again there are two possibilities, namely,

(1) The internal symmetry is broken but the supersymmetry is not. This happens when $ed<0$ because then the potential minimum occurs for D(A)=0, $|A|^2 = |d/e|$.

(2) The supersymmetry is broken but the internal symmetry is not. This happens when $ed>0$ because then the potential minimum occurs for

$$A = 0 \,, \qquad |D(A)| = |D(0)| = |d| \neq 0 \,. \quad (3.29)$$

Of course, in more complicated models there could be a spontaneous breakdown of both internal and super symmetry. But for simplicity we shall consider only the cases (1) and (2). We shall also assume for simplicity that the gauge-group is U(1) and that there is only one matter-field.

5. Spontaneous Breakdown of Internal Group Symmetry.

Let us first consider the case $ed<0$ when the internal symmetry is broken. For simplicity, and because it is the most interesting case, we shall assume that the gauge - and matter-field fermions λ and χ have opposite chirality. Note that when χ has definite chirality the mass-terms and Yukawa couplings vanish, so the vanishing of F(A) is automatic and does not have to be assumed. In that case the Lagrangian (3.27) reduces to

$$\mathcal{L}^{gauge}_{ss} = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\lambda}\not{\partial}\lambda + \bar{\chi}\not{\partial}\chi + e\bar{\chi}A\lambda + \tfrac{1}{2}|D_\mu A|^2 + \tfrac{1}{2}|d + e|A|^2|^2 \,. \quad (3.30)$$

Since the potential $|d + e|A|^2|^2$ in (3.30) is just the standard Higgs potential, with minimum at $\langle A\rangle^2 = |d/e|$ one sees that the non-supersymmetric part of the Lagrangian undergoes the usual Higgs mechanism

$$\mathcal{L}^{gauge}_{ss} \Rightarrow \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}|D_\mu A|^2 + \tfrac{m^2}{2}A_\mu^2 + \text{interaction terms}, \quad (3.31)$$

in which the gauge-field acquires a mass $m = e|d/e|^{1/2} = |ed|^{1/2}$. However, in addition to this conventional Higgs mechanism one sees from (3.30) that the fermion terms also undergo a Higgs mechanism, namely,

$$e\bar{\chi}A\lambda \rightarrow e\bar{\chi}\lambda|\mathring{A}| + \ldots \quad = m\bar{\chi}\lambda + \ldots \quad , \quad (3.32)$$

where .... denotes interaction terms. Furthermore, if one introduces the Dirac field $\psi = \chi + \lambda$ of Section and recalls that $\gamma_5\gamma_\mu$ and γ_0 connect the same and opposite chiralities respectively, one sees that the mechanism (3.32) can be written more fully as

$$\bar{\lambda}\not{\partial}\lambda + \bar{\chi}\not{\partial}\chi + e\bar{\chi}A\lambda \equiv \bar{\psi}(\not{\partial} + eA)\psi = \bar{\psi}(\not{\partial} + m + eB)\psi, \quad (3.33)$$

where $B = A - \langle A\rangle$. Thus the Higgs mechanism for the fermions combines the gauge-field and matter-field fermions into a Dirac fermion with the same mass as the gauge-field.

Finally one notes that since

$$-\frac{1}{2}\left|d+e|A|^2\right|^2 = \frac{1}{2}\left|2e|\lambda|B+e|B|^2\right|^2 = \frac{m}{2}|B|^2 + O(B^3)\ , \tag{3.34}$$

the scalar field B also acquires the mass $m = |ed|^{\frac{1}{2}}$. So the gauge-superfield and the matter-superfield have merged to form a single superfield

$$\tilde{\Phi} = \{B, \psi, A_\mu\}\ , \tag{3.35}$$

consisting of a real scalar B, a real vector A_μ, and a Dirac field ψ, all with the same mass $m = |ed|^{1/2}$.

6. Spontaneous Breakdown of Gauge Supersymmetry (D-breaking)

Let us now consider Case (2) in which the supersymmetry is broken by D(A) but the internal symmetry is not. We shall not assume here that the fermion fields have definite chirality or that F(A) is zero (although we shall assume that F(A) does not cause any symmetry breakdown). Then the U(1) supersymmetric gauge Lagrangian is

$$\mathcal{L}^{gauge}_{ss}(x) = \frac{F^2}{4} + \bar{\lambda}\ \lambda + \bar{\chi}(\partial\!\!\!/ + m + gA)\chi + \frac{1}{2}|D_\mu A|^2 + \frac{1}{2}|F(A)|^2 + \frac{1}{2}\left|d+e|A|^2\right|^2 . \tag{3.36}$$

The nice thing about this case is that, since the potential minimum occurs at A=0, there is no need to shift the scalar field, so the mass-spectrum can be read directly from (3.36), and is

$$m^2_\chi = m^2 \qquad m^2_A = m^2 + 2ed \qquad\qquad m^2(\lambda) = m^2(A_\mu) = 0\ . \tag{3.37}$$

Hence the masses of the gauge supermultiplet remain degenerate (and zero) but the masses of the matter-multiplet split. Thus the gauge-supermultiplet acts as a catalyst for the breaking of the matter-superfield, but is not itself broken. Note that in this case there is no sum-rule such as (3.23) which is preserved after the supersymmetry breakdown.

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